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On the Dimension of Objects and Categories II. Finite Ordered Sets*

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In this paper we consider the problem of finding the global dimension of the functor category $\mathcal{A}^{\Pi}$ where $\mathcal{A}$ is an Abelian category and Π is a finite partially-ordered set. Some advance has been made in this direction since the writing of [*11*, Chapter IX], and in particular the condition that $\mathcal{A}$ have enough projectives has been eliminated. Nevertheless, a complete solution to the problem is still lacking. In order to understand the present treatment, the reader should know the basic theory of adjoint functors and the Ext functor as found, for example, in [*11*, Chapters V and VII], and he must also have read Section 1 of [*12*].

In Section 5 we make a brief review of the notion of proper (h.f.) classes of exact sequences, and we shall indicate how our previous results can be generalized to the relative situation. The final section contains a discussion of functors which preserve Ext. It is unrelated to the work on global dimension.

Our notation and terminology will be as in [*12*]. In particular, $\mathcal{A}$ and $\mathcal{B}$ will always denote Abelian categories, and the word "coadjoint" will be used in place of "left adjoint."

1. ORDERED SETS

An *ordered set* is a small category Π such that for any pair of objects $i, j \in \Pi$, the set of morphisms from i to j is either empty or has precisely one element. In the latter case we write $i \leqslant j$. Thus we have $i \leqslant i$ for all $i \in \Pi$ (since every object has its identity morphism), and if $i \leqslant j$ and $j \leqslant k$, then $i \leqslant k$ (since composition must be defined). If $i \leqslant j$ and $i \neq j$, then we write $i < j$. If neither $i \leqslant j$ nor $j \leqslant i$, then we say that i and j are ***incompatible.*** On the other hand, if both $i \leqslant j$ and $j \leqslant i$, then i and j are isomorphic objects in Π.

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Let Π' be a full subcategory of Π obtained by taking one member from each isomorphism class of objects. Then Π' is an equivalent subcategory of Π [*11*, p. 52], and it follows that the functor categories $\mathfrak{A}^{\Pi}$ and $\mathfrak{A}^{\Pi'}$ are equivalent. The category Π' is an ordered set with the property that any two members which precede each other are equal. Such an ordered set will be called *skeletal.*

Let Π be an ordered set (not necessarily skeletal), and let D be an object of $\mathfrak{A}^{\Pi}$. For $i \in \Pi$, let $T_i : \mathfrak{A}^{\Pi} \to \mathfrak{A}$ be the corresponding evaluation functor, and for $i \leqslant j$, let $T_{ij} : T_i \to T_j$ denote the corresponding natural transformation. Then we define $D_i = T_i(D)$ and $D_{ij} = T_{ij}(D)$. Also if $f : D \to D'$ is a morphism in $\mathfrak{A}^{\Pi}$, we define $f_i = T_i(f)$. The coadjoint $S_i : \mathfrak{A} \to \mathfrak{A}^{\Pi}$ for T_i is given by

$$
\begin{aligned}
S_i(A)_j &= A \quad \text{for} \quad i \leqslant j \\
&= 0 \quad \text{otherwise} \\
S_i(A)_{jk} &= 1_A \quad \text{for} \quad i \leqslant j \leqslant k \\
&= 0 \quad \text{otherwise} \\
S_i(\alpha)_j &= \alpha \quad \text{for} \quad i \leqslant j \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

If $\mathfrak{A}$ is the category of right R-modules and Π is finite, then

$$\bigoplus_{i \in \Pi} S_i(R)$$

is a small projective generator for $\mathfrak{A}^{\Pi}$, and consequently $\mathfrak{A}^{\Pi}$ is equivalent to the category of right modules over the ring of endomorphisms of this object [*11*, p. 104, Theorem 4.1]. Such an endomorphism is determined by a $\Pi \times \Pi$ matrix (α_{ij}) where $\alpha_{ij} \in [S_j(R), S_i(R)]$. By adjointness the latter is isomorphic to

$$
\begin{aligned}
[R, T_j S_i(R)] &= [R, R] \approx R \quad \text{for} \quad i \leqslant j \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$

Thus we see that $\mathfrak{A}^{\Pi}$ is equivalent to the category of right modules over the subring $R(\Pi)$ of the full ring of $\Pi \times \Pi$ matrices over R consisting of all those matrices (r_{ij}) such that r_{ij} is an arbitrary element of R for $i \leqslant j$, and $r_{ij} = 0$ otherwise. It follows that any ring of the form $R(\Pi)$ has its category of right modules equivalent to the category of right modules over $R(\Pi')$ for some skeletal ordered set Π'.

If $\mathfrak{A}$ is the category of left R-modules, or in other words the category of right R^*-modules, then $\mathfrak{A}^{\Pi}$ is equivalent to the category of right modules over $R^*(\Pi)$, or in other words the category of left modules over $(R^*(\Pi))^*$, and the latter is the same as $R(\Pi^*)$.

Until Section 5, Π will always denote a finite, skeletal ordered set. If $i < j$ and for no k do we have $i < k < j$, then we call j a *cover* for i and i a *cocover* for j. Observe that any two covers (cocovers) of an element are necessarily incompatible.

If $i < j$, we can find a sequence

$$i < i_1 < i_2 < \cdots < i_m < j$$

where each term is a cover for the preceding one. In particular, it follows that if i has only one cover i_1 and $i < j$, then $i_1 \leqslant j$. A dual remark applies to cocovers.

An element i is called *maximal* in Π if for no j in Π do we have $i < j$, and it is called *minimal* in Π if it is maximal in Π^*. If Π has just one maximal member, we say that Π is *terminal.* If Π has just one minimal member, we say Π is *initial.*

The set Π can be represented diagrammatically as follows. First define the *depth* of an element i to be the greatest integer m such that there is a sequence

$$i_1 < i_2 < \cdots < i_m = i.$$

Put the elements of Π in rows, the elements of depth m going in the mth row, and draw a line between two elements if and only if one is a cover for the other. (Such a line may have to skip rows). Now if $i < j$, then the depth of i is necessarily less than the depth of j, and from the sequence (1) we see that $i < j$ is equivalent to having a broken path starting at i and ending at j and continually descending.

An *ordered subset* of Π is a *full* subcategory of Π. If Π' is an ordered subset of Π, let

$$F : \mathcal{A}^{\Pi} \to \mathcal{A}^{\Pi'}$$

denote the restriction functor. For $i \in \Pi$, let

$$\Pi'_i = \{i' \in \Pi' \mid i' \leqslant i\}.$$

We define a functor (which we shall refer to as the *extension* functor with respect to the subset Π')

$$G : \mathcal{A}^{\Pi'} \to \mathcal{A}^{\Pi}$$

by taking $G(D')_i$ to be the colimit of the restriction of D' to Π'_i. Since $\mathcal{A}$ is Abelian and Π is finite, the colimit always exists. If $i \leqslant j$ in Π, then $\Pi'_i \subset \Pi'_j$, and consequently there is an induced morphism $G(D')_{ij} : G(D')_i \to G(D')_j$ of colimits. Likewise if $f : D' \to D''$ in $\mathcal{A}^{\Pi'}$, then there are induced morphisms $G(f)_i : G(D')_i \to G(D'')_i$ which define a morphism $G(f)$ in $\mathcal{A}^{\Pi}$. We leave it to the reader to verify that G is a coadjoint for F. (G is just a special case of the Kan construction). When Π' is a single element i, the functor G is the functor S_i defined above. Clearly the composition FG is the identity functor on $\mathcal{A}^{\Pi'}$, and it follows that F is representative.

Each ordered set Π is divided into equivalence classes, called *components*, by the equivalence relation generated by the order relation. If Π' is an ordered subset of Π such that for each $i \in \Pi$ the components of Π'_i are terminal, then Π' is called a *separated* subset of Π. In this case the colimit of the restriction of D' to Π'_i is the coproduct of the objects D'_j over the maximal members j of the components of Π'_i. Consequently, in this case G is exact. By [*12*, Corollary 1.2] we thus obtain

LEMMA 1.1. *If Π' is a separated subset of Π, then for all $D' \in \mathcal{A}^{\Pi'}$ we have*

$$\text{h.d.}_{\Pi'} D' = \text{h.d.}_{\Pi} G(D').$$

Consequently

$$\text{gl.dim. } \mathcal{A}^{\Pi'} \leqslant \text{gl.dim. } \mathcal{A}^{\Pi}. \tag{2}$$

Dually we say that Π' is a *coseparated* subset of Π if Π'^* is a separated subset of Π^*. Using the relations

$$(\mathcal{A}^{\Pi})^* = \mathcal{A}^{*\Pi^*}, \qquad \text{gl. dim. } \mathcal{A} = \text{gl. dim. } \mathcal{A}^*, \tag{3}$$

we obtain

COROLLARY 1.2. *If Π' is a coseparated subset of Π, then* (2) *is valid.*

Remark. The inequality (2) is not valid for general subsets $\Pi' \subset \Pi$. For example, let Π and Π' be given by

Π $\qquad$ Π' $\qquad$ (4)

Then Π' is an ordered subset of Π, but as we shall see presently, the global dimension of $\mathcal{A}^{\Pi'}$ is one greater than that of $\mathcal{A}^{\Pi}$.

An ordered subset $\Pi' \subset \Pi$ is called *final* if for each $i' \in \Pi'$, the relation $i' \leqslant j$ implies $j \in \Pi'$. Clearly a final subset is separated, and the functor G in this case extends an object $D' \in \mathcal{A}^{\Pi'}$ by putting 0 at i for all $i \notin \Pi'$.

2. An Upper Bound for gl.dim. $\mathfrak{A}^\Pi$

If Π is nonempty, then taking Π' to consist of a single element in Lemma 1.1, we see that gl.dim. $\mathfrak{A} \leqslant$ gl.dim. $\mathfrak{A}^\Pi$. In particular, if $\mathfrak{A}$ has infinite global dimension, then so does $\mathfrak{A}^\Pi$. Consequently, the only interesting case is where $\mathfrak{A}$ has finite global dimension, and we shall always assume this to be the case in the sequel.

We define the *rank* of an element $i \in \Pi$ as follows: First we agree that all minimal elements are to have rank 0, and all other elements are to have rank $\geqslant 1$. Inductively we define i to have rank $\geqslant r$ ($r \geqslant 2$) if it is preceded by a pair of incompatible elements of rank $\geqslant r - 1$. Rank i is then defined as the greatest integer r such that i has rank $\geqslant r$. Clearly if $i \leqslant j$, then rank $i \leqslant$ rank j. Also if rank $i = r$, then the set of elements preceding i which also have rank r is linearly ordered. Otherwise there would exist two incompatible such elements, and consequently i would have rank $\geqslant r + 1$.

Consider an object $D \in \mathfrak{A}^\Pi$. For each $i \in \Pi$, the identity morphism on D_i extends to a unique morphism $S_i(D_i) \to D$, and these are the coordinates of an epimorphism

$$\bigoplus_{i \in \Pi} S_i(D_i) \to D. \tag{1}$$

If we denote the coproduct by D^0 and the kernel by K^1 in (1), then we have an exact sequence

$$0 \to K^1 \to D^0 \to D \to 0 \tag{2}$$

where $K_i^1 = 0$ for all elements i of rank 0 (minimal elements). Now $D_i^0 \to D_i$ is a retraction for all i, hence $K_i^1 \to D_i^0$ is a coretraction. The morphism $D_i^0 \to D_j^0$ is also a coretraction for $i \leqslant j$, and so since

$$K_i^1 \to K_j^1 \to D_j^0 = K_i^1 \to D_i^0 \to D_j^0,$$

we see that K_{ij}^1 is a coretraction. For each element i of rank 1 which is preceded by some other element of rank 1, let $k(i)$ be the (unique) maximal such element. Let L_i^1 be such that

$$K_i^1 = L_i^1 \oplus \overline{\mathrm{Im}(K_{k(i),i}^1)}.$$

For all other elements i define $L_i^1 = K_i^1$. Then we have an exact sequence

$$0 \to K^2 \to \bigoplus_{i \in I} S_i(L_i^1)$$

where $K_i^2 = 0$ for all elements i of rank $\leqslant 1$. We can then proceed inductively to produce exact sequences

$$0 \to K^{r+1} \to \bigoplus S_i(L_i^r) \to K^r \to 0 \tag{3}$$

where $K_i^{r+1} = 0$ for all elements i of rank $\leqslant r$.

LEMMA 2.1. *Let $D \in \mathfrak{A}^\Pi$ be such that* $\text{h.d.}_{\mathfrak{A}} D_i \leqslant n$ *for all* $i \in \Pi$, *and suppose that the maximum rank of an element in Π is m. Then* $\text{h.d.}_\Pi D \leqslant m + n$, *and consequently*

$$\text{gl.dim. } \mathfrak{A}^\Pi \leqslant m + \text{gl.dim. } \mathfrak{A}.$$

Proof. Letting $K^0 = D$, we have the exact sequences (3) for all $r \geqslant 0$, and furthermore $K^{m+1} = 0$. Now for each i and each r, L_i^r is a retract of K_i^r, and consequently $\text{h.d.}_{\mathfrak{A}} L_i^r \leqslant \text{h.d.}_{\mathfrak{A}} K_i^r$. But since for each j, K_j^{r+1} is a retract of a finite coproduct of objects of the form L_i^r, it follows that

$$\text{h.d.}_{\mathfrak{A}} K_j^{r+1} \leqslant \max_{i \in \Pi} \text{h.d.}_{\mathfrak{A}} (K_i^r).$$

Therefore we see inductively that $\text{h.d.}_{\mathfrak{A}} L_i^r \leqslant n$ for all i and all r. Using Lemma 1.1, it follows that the middle term in each of the exact sequences (3) has homological dimension $\leqslant n$, and so these sequences yield successively

$$\begin{aligned} &\text{h.d.}_\Pi K^m \leqslant n \\ &\text{h.d.}_\Pi K^{m-1} \leqslant n + 1 \\ &\qquad \vdots \\ &\text{h.d.}_\Pi K^1 \leqslant n + m - 1 \\ &\text{h.d.}_\Pi K^0 \leqslant n + m \end{aligned}$$

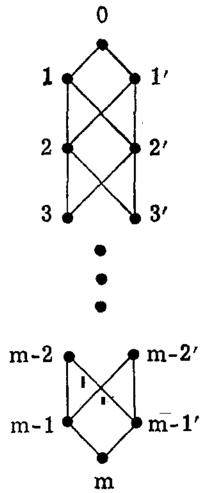
as required.

Remark. The inequality in Lemma 2.1 may be strict. For example, let Π be a linearly-ordered set of three elements. Then the maximum rank of an element is one, and so by two applications of Lemma 2.1 we have

$$\text{gl.dim. } \mathfrak{A}^{\Pi \times \Pi} = \text{gl.dim.} (\mathfrak{A}^\Pi)^\Pi \leqslant 1 + \text{gl.dim. } \mathfrak{A}^\Pi \leqslant 2 + \text{gl.dim. } \mathfrak{A}.$$

Nevertheless, $\Pi \times \Pi$ has an element of rank 3.

The definition of rank requires that if Π has an element of rank $r > 1$, then Π has at least two elements of rank $r - 1$. Thus we see that the least number of elements Π can have in order to have an element of rank m is $2m$, in which case Π is the following.



(4)

In this case we call Π the *m-braid*, and we denote it by β_m. Thus β_0 is a single point, β_1 is the linearly-ordered set of two elements, β_2 is the square, and β_3 is given by the second of the diagrams (4) of Section 1. By Lemma 1.1 we have

$$\text{gl. dim. } \mathfrak{A}^{\beta_m} \leqslant m + \text{gl. dim. } \mathfrak{A}. \tag{5}$$

In Theorem 3.6 we shall prove equality.

3. Muscles

For $i \in \Pi$ and $A \in \mathfrak{A}$, let $L_i(A)$ denote the object in $\mathfrak{A}^\Pi$ with A at i and 0 elsewhere.

Lemma 3.1. *If Π is nonempty, then there exists an element $i \in \Pi$ and an object $A \in \mathfrak{A}$ such that*

$$\text{gl.dim. } \mathfrak{A}^\Pi = \text{h.d.}_\Pi L_i(A). \tag{1}$$

Proof. By induction on the number of elements in Π. If this number is one, then we simply take A to have maximum homological dimension in $\mathfrak{A}$. Otherwise let j be any minimal element of Π and let $\Pi' = \Pi - \{j\}$. Then Π' is a final subset of Π. Let F and G denote the restriction and extension functors. Suppose that D is an object of $\mathfrak{A}^\Pi$ of maximum homological dimension n. Then we have an exact sequence in $\mathfrak{A}^\Pi$

$$0 \to K \to D \to L_j T_j(D) \to 0$$

where $K_j = 0$. If $\text{h.d.}_\Pi L_j T_j(D) < n$, then we must have $\text{h.d.}_\Pi K = n$. Now $K = GF(K)$, and so by Lemma 1.1 we see that $\text{h.d.}_{\Pi'} F(K) = n$. Again by Lemma 1.1 this implies that $\text{gl.dim. } \mathfrak{A}^{\Pi'} = \text{gl.dim. } \mathfrak{A}^{\Pi}$, and so by induction we may find $A \in \mathfrak{A}$ and $i \in \Pi'$ such that $\text{h.d.}_{\Pi'} L_i'(A) = n$. Since $GL_i'(A) = L_i(A)$, a final application of Lemma 1.1 shows that $\text{h.d.}_\Pi L_i(A) = n$.

Remark. There is no guarantee that the object A of (1) is one of maximum homological dimension in $\mathfrak{A}$. Nor is there any guarantee that the i of that equation can be found independently of the category $\mathfrak{A}$.

For $i \in \Pi$, let ${}_i\Pi = \{j \in \Pi \mid i \leqslant j\}$. Then ${}_i\Pi$ is a final subset of Π, and letting F_i denote the restriction functor we have, using Lemma 1.1,

$$\text{h.d.}_{{}_i\Pi} F_i L_i(A) = \text{h.d.}_\Pi L_i(A). \tag{2}$$

Therefore it follows from (1) that

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = \max_{i \in \Pi} \text{gl.dim. } \mathfrak{A}^{{}_i\Pi}$$

Consequently, if we denote $\Pi_k = \{j \in \Pi \mid j \leqslant k\}$ and use Eq. (3) of Section 1, we obtain

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = \max_{k \in \Pi} \text{gl.dim. } \mathfrak{A}^{\Pi_k}. \tag{3}$$

For $i \leqslant k$, let ${}_i\Pi_k = \{j \in \Pi \mid i \leqslant j \leqslant k\}$, and for $A \in \mathfrak{A}$, let us define the *A-muscle of Π between i and k* as the object in $\mathfrak{A}^{{}_i\Pi_k}$ with A at i and 0 elsewhere. Then using (3), as well as (1) and (2) applied to Π_k, we obtain

COROLLARY 3.2. *The global dimension of $\mathfrak{A}^{\Pi}$ is the maximum of the homological dimensions of all of its muscles.*

An object $D \in \mathfrak{A}^{\Pi}$ will be called *split* if the inclusion morphism

$$\bigcup_{i<q} \text{Im } D_{iq} \to D_q$$

is a coretraction for each $q \in \Pi$. If we denote the union by D_q', we can then write $D_q = D_q' \oplus D_q''$ for some subobject $D_q'' \subset D_q$. An easy induction on the depth of q shows that

$$D_q' = \bigcup_{i<q} D_{iq}(D_i'')$$

for all q, and consequently we get an exact sequence

$$0 \to K \xrightarrow{\partial} \bigoplus_{i \in \Pi} S_i(D_i'') \to D \to 0. \tag{4}$$

Evaluating (4) at an element $q \in \Pi$, we obtain

$$0 \longrightarrow K_q \xrightarrow{\delta_q} \bigoplus_{i \leqslant q} D_i'' \longrightarrow D_q' \oplus D_q'' \longrightarrow 0, \tag{5}$$

where the image of δ_q is in $\bigoplus_{i<q} D_i''$. Consequently if we denote the injections and projections for $\bigoplus_{i \leqslant q} D_i''$ by u_{iq} and p_{iq} respectively, then we have

$$\delta_q = \sum_{i<q} u_{iq} p_{iq} \delta_q \,. \tag{6}$$

Now if E denotes the middle term in (4), then by definition of E we have $u_{iq} = E_{iq} u_{ii}$ for $i < q$. Thus if we write $u_{ii} p_{iq} \delta_q = \tau_i$ for $i < q$, then (6) can be rewritten

$$\delta_q = \sum_{i<q} E_{iq} \tau_i \,. \tag{7}$$

Let $\mathcal{G}$ denote the category of finitely-generated Abelian groups. We shall say that an object $D \in \mathcal{G}^\Pi$ is *pointwise free* if D_i is a free Abelian group (necessarily of finite rank) for each $i \in \Pi$. If D is split and pointwise free, then the D_i'', being subgroups of the D_i, are free, and consequently the object K of the exact sequence (4) is pointwise free. We shall say tentatively that D is *projective* if $K = 0$ relative to all choices of complements D_i''. (It will result from the remark following Lemma 3.3 that if this is true for one such choice of complements, then it is true for them all, and furthermore that the above definition of "projective" agrees with the usual one).

Now if $\mathfrak{A}$ is any Abelian category and $A \in \mathfrak{A}$, then we have the functor $A \otimes \quad : \mathcal{G} \to \mathfrak{A}$ which can be described as the unique cokernel preserving functor such that $A \otimes Z = A$. An object $D \in \mathcal{G}^\Pi$ gives rise to the object $A \otimes D \in \mathfrak{A}^\Pi$ defined as the composition of D with $A \otimes \quad$.

LEMMA 3.3. *If D is split and pointwise free but not projective, then for each nonzero object $A \in \mathfrak{A}$ we have*

$$\text{h.d.}_\Pi\, A \otimes D \geqslant 1 + \text{h.d.}_\mathfrak{A}\, A. \tag{8}$$

Proof. Letting K be as in the exact sequence (4), let q be minimal with respect to the property that $K_q \neq 0$. Since (4) splits pointwise, we obtain an exact sequence in $\mathfrak{A}^\Pi$ of the form

$$0 \to A \otimes K \to A \otimes E \to A \otimes D \to 0.$$

In the notation of Lemma 1.7 of [*12*], let us denote $L(A) = A \otimes K$. Then applying $A \otimes \quad$ to (7), that lemma yields the desired inequality (8).

Remark. In $\mathscr{G}^{\Pi}$ we know that any object of the form

$$\bigoplus_{i \in \Pi} S_i(Z^{n_i}) \tag{9}$$

is projective in the usual sense. On the other hand, if D is a projective in $\mathscr{G}^{\Pi}$ in the usual sense, then D is a retract of an object of the form (9), and since the latter is split and pointwise free, the same is true of D. Now if relative to some choice of complements D_i'' we had $K \neq 0$, then taking $A = Z$ in (8), we see that h.d. $D \geqslant 1$, contradicting the fact that D is projective. Thus we see that the projectives in $\mathscr{G}^{\Pi}$ are precisely the objects of the form (9). (It is true more generally that any retract of an object of the form $\bigoplus_{i \in \Pi} S_i(A_i)$ in $\mathfrak{A}^{\Pi}$ [$\mathfrak{A}$ an arbitrary Abelian category] is of the same form [*11*, p. 227]. However we shall not be using this latter fact). It follows from this characterization of projectives, together with the fact that $K = 0$ for all choices of complements if it is so for one of them, that an object in $\mathscr{G}^{\Pi}$ is projective if and only if its restriction to Π_i is projective for all $i \in \Pi$.

THEOREM 3.4. *Let D be pointwise free, and consider exact sequences in* $\mathscr{G}^{\Pi}$

$$\begin{gathered} 0 \to K^1 \to \bigoplus_{i \in \Pi} S_i(F_{0i}) \to D \to 0 \\ 0 \to K^2 \to \bigoplus_{i \in \Pi} S_i(F_{1i}) \to K^1 \to 0 \\ \vdots \\ 0 \to K^{m-1} \to \bigoplus_{i \in \Pi} S_i(F_{m-2,i}) \to K^{m-2} \to 0 \end{gathered} \tag{10}$$

where F_{ki} is free for $0 \leqslant k \leqslant m-2$. Suppose that h.d.$_{\mathfrak{A}}\, A = n$:

(a) *If* h.d.$_{\Pi}\, K^{m-1} = 1$, *then* h.d.$_{\Pi}\, A \otimes D \leqslant m + n$.

(b) *If K^{m-1} is split but not projective, then* h.d.$_{\Pi}\, A \otimes D \geqslant m + n$.

Proof. (a) If h.d.$_{\Pi}\, K^{m-1} = 1$, then we can find as exact sequence

$$0 \to \bigoplus_{i \in \Pi} S_i(F_i) \to \bigoplus_{i \in \Pi} S_i(F_{m-1,i}) \to K^{m-1} \to 0.$$

Tensoring this last sequence and the sequences (10) with A, the result follows easily from Lemma 1.1 using the fact that $S_i(Z) \otimes A = S_i(A)$.

(b) If K^{m-1} is split but not projective, then the result follows by using Lemma 3.3 and the tensored sequences (10).

Let D be a Z-muscle of homological dimension m. We shall say that D is *strong* if in the projective resolution (10), the object K^{m-1} is split. To see that this notion is independent of projective resolutions, consider two resolutions

$$0 \to K^{m-1} \to P^{m-2} \to \cdots \to P^1 \to P^0 \to D \to 0$$

$$0 \to L^{m-1} \to Q^{m-2} \to \cdots \to Q^1 \to Q^0 \to D \to 0$$

with the P's and Q's projective. Then by a theorem of Schanuel (see [9]) we have $K^{m-1} \oplus P \approx L^{m-1} \oplus Q$ for some projectives P and Q. It follows that if K^{m-1} is split, then L^{m-1}, being a retraction of a split object, is itself split.

Combining Corollary 3.2 and Theorem 3.4, we now obtain

COROLLARY 3.5. *Let m be the maximum homological dimension for all Z muscles of Π. Then for any nontrivial $\mathcal{A}$ we have*

$$\text{gl.dim. } \mathcal{A}^{\Pi} \leqslant m + \text{gl.dim. } \mathcal{A}.$$

Furthermore, if there is a strong Z-muscle of homological dimension m, then we have equality.

Suppose that we can write Π as a disjoint union

$$\bigcup_{k=0}^{m} \Pi^{(k)},$$

and let n_k be the number of elements in $\Pi^{(k)}$. Assume that $n_0 = n_m = 1$, and $n_k > 1$ for $1 \leqslant k \leqslant m - 1$. Suppose that $\Pi^{(k+1)}$ is the set of covers of each element in $\Pi^{(k)} (0 \leqslant k \leqslant m - 1)$. Then we shall call Π a *generalized m-braid.* (If $n_k = 2$ for $1 \leqslant k \leqslant m - 1$, then Π is the ordinary m-braid). Notice that $\Pi^{(k)}$ is the set of elements of rank k. For $1 \leqslant k \leqslant m$, let $M_k(A)$ be the object of $\mathcal{G}^{\Pi}$ defined by

$$\begin{aligned} M_k(A)_i &= A \quad \text{if rank } i \geqslant k \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where the morphisms in $M_k(A)$ are 1_A whenever possible. Then we have an exact sequence

$$0 \to M_1(A) \to S_0(A) \to L_0(A) \to 0, \tag{11}$$

where as usual $L_0(A)$ denotes the object of $\mathcal{A}^{\Pi}$ with A at vertex 0 and zeroes elsewhere. Also for $1 \leqslant k \leqslant m - 1$ we have exact sequences

$$0 \to M_{k+1}(A^{n_k-1}) \to \bigoplus_{i \in \Pi^{(k)}} S_i(A) \to M_k(A) \to 0. \tag{12}$$

Now if $A \in \mathcal{G}$ and $A \neq 0$, then clearly $M_{m-1}(A)$ is split but not projective, and so using the fact that $n_k - 1 \geqslant 1$ for $1 \leqslant k \leqslant m - 1$, a judicious choice of A's in (11) and (12) shows that $L_0(Z)$ is a strong Z-muscle of homological dimension m. Consequently using Corollary 3.5 and Lemma 2.1, we obtain

THEOREM 3.6. *If Π is a generalized m-braid, then*

$$\text{gl.dim. } \mathcal{A}^{\Pi} = m + \text{gl.dim. } \mathcal{A}$$

for all nontrivial $\mathcal{A}$.

LEMMA 3.7. *Let Π' be a final subset of Π, and let D be the object of $\mathcal{G}^{\Pi}$ such that $D_i = Z$ for $i \in \Pi'$, $D_i = 0$ for $i \notin \Pi'$, and D_{ij} is the identity function on Z wherever possible. Let M denote the set of minimal elements of Π', and consider the exact sequence in $\mathcal{G}^{\Pi}$*

$$0 \to K \xrightarrow{\beta} \bigoplus_{m \in M} S_m(Z) \xrightarrow{\alpha} D \to 0$$

where for each m the mth *coordinate of α is induced by the identity function on Z. Then K is a split object such that $K_m = 0$ for all $m \in M$.*

Proof. Given $\mu \in \Pi'$, let $1, 2, \ldots, n$ denote those elements of M which are less than or equal to μ. Then α_μ is the codiagonal morphism $Z^n \to Z$, and so we can take as β_μ the morphism $Z^{n-1} \to Z^n$ defined by

$$\beta_\mu(x_1, \ldots, x_{n-1}) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{n-1} - x_{n-2}, - x_{n-1}).$$

If $\lambda \leqslant \mu$ and if $i_1, i_2, \ldots, i_k$ denote in order those elements among $1, 2, \ldots, n$ which are $\leqslant \lambda$, then the morphism

$$K_{\lambda\mu} : Z^{k-1} \to Z^{n-1}$$

is given by the matrix whose tth column $(1 \leqslant t \leqslant k - 1)$ has 1 in rows $i_t, i_t + 1, \ldots, i_{t+1} - 1$, and 0 elsewhere. Thus the morphism $\bigoplus_{\lambda<\mu} K_\lambda \to K_\mu$ is determined by a matrix whose columns consist of solid blocks of 1's with 0's elsewhere. It suffices now to show that such a matrix has a diagonal form consisting of 1's and 0's. Consider the first row which contains a 1, and among all columns having 1 in this row, subtract one which contains the least number of 1's from the others. Then reduce all but one of the 1's in this column to 0, and apply induction.

A pair of nonempty subsets I, J of Π will be said to constitute a *crown* in Π if each element of I precedes (that is, is less than or equal to) at least two members of J, and each member of J follows (is greater than or equal to) at least two members of I. An element $k \in \Pi$ is a *crosspoint* for the crown if it precedes at least two members of J and follows at least two members of I.

Notice that if k is a crosspoint which is not in J and has only one cover, then the cover is also a crosspoint. A similar remark applies to cocovers. The crown is called *uncrossed* if it has no crosspoints. In this case it follows that any pair of elements of I are incompatible, for otherwise the greater of the two would be a crosspoint. Similarly for the J's. Also we see in this case that no member of I follows a member of J, for otherwise it would result from the fact that each member of I has two members of J following it that a pair of elements of J would be compatible.

A *suspended* crown is a crown together with an element which precedes each member of I and an element which follows each member of J.

THEOREM 3.8. *Let Π be an uncrossed, suspended crown with minimal element* 0. *Then $L_0(Z)$ is a strong Z-muscle in $\mathcal{G}^\Pi$ of homological dimension* 3. *Consequently*

$$\text{gl.dim. } \mathcal{A}^\Pi = 3 + \text{gl.dim. } \mathcal{A}$$

for all nontrivial Abelian categories $\mathcal{A}$.

Proof. Let $1, 2,..., n$ and $1', 2',..., m'$ denote the elements of I and J respectively, and let μ denote the maximal element. Then we have an exact sequence

$$0 \to D \to S_0(Z) \to L_0(Z) \to 0$$

where D is as in Lemma 3.7. Forming the exact sequence

$$0 \to K \to \bigoplus_{i=1}^{n} S_i(Z) \to D \to 0$$

of that lemma, we know that K is split. Now if K were projective, then since K has nonzero objects only at μ and the mutually incompatible elements i', it would follow that the morphism

$$\bigoplus_{i=1}^{m} K_{i'} \to K_\mu \tag{13}$$

is a split monomorphism. This would mean that the number of generators of the domain of (13) is no greater than the number of generators of the range. To see that this is not the case, let k_i denote the number of elements of I preceding i' for $1 \leqslant i \leqslant m$. Then the number of generators of the domain of (13) is

$$\sum_{i=1}^{m} (k_i - 1) = \sum_{i=1}^{m} k_i - m.$$

Since each element of I has at least two members of J following it and each member of J has at least two members of I preceding it, we find

$$\sum_{i=1}^{m} k_i - m \geqslant 2 \max(m, n) - m > n.$$

Since K_μ has only $n - 1$ generators, this completes the proof of the lemma.

4. Dimension for Finite Ordered Sets

One would like to show that for each finite ordered set Π, there is an integer m such that

$$\text{gl.dim. } \mathcal{A}^{\Pi} = m + \text{gl.dim. } \mathcal{A} \tag{1}$$

for all nontrivial $\mathcal{A}$. By Corollary 3.5 it would suffice to show that all Z-muscles are strong. This may not be true, but a counterexample will have to involve a set with an element of rank at least 4. To see this, suppose that $L_0(Z)$ is a Z-muscle of Π where all elements of Π have rank $\leqslant 3$. By Lemma 2.1, $L_0(Z)$ must have homological dimension $\leqslant 3$, and so it follows from 3.7 that $L_0(Z)$ is strong.

In any case, let us define the *dimension* of Π to be the integer m (if it exists) satisfying (1) for all nontrivial Abelian categories $\mathcal{A}$. From the isomorphism of categories

$$\mathcal{A}^{\Pi_1 \times \Pi_2} \approx (\mathcal{A}^{\Pi_1})^{\Pi_2},$$

it follows that

$$\text{dim. } \Pi_1 \times \Pi_2 = \text{dim. } \Pi_1 + \text{dim. } \Pi_2$$

providing the right side is defined. In particular if we define the *m-cube* to be the product (of categories) of m copies of the 1-braid β_1, then we see that its dimension is m. (Notice that the 2-cube is the 2-braid, and the 3-cube is an uncrossed, suspended crown.) Also it follows from Eq. (3) of Section 1 that

$$\dim \Pi^* = \dim \Pi$$

providing the right side is defined.

We now proceed to characterize ordered sets of dimensions 0, 1, and 2.

Lemma 4.1. *If $\mathcal{A}$ is nontrivial (of finite global dimension), then*

$$\text{gl.dim. } \mathcal{A}^{\Pi} = \text{gl.dim. } \mathcal{A} \tag{2}$$

if and only if Π does not contain β_1 as an ordered subset (or equivalently, every pair of distinct elements in Π are incompatible).

Proof. If Π does not contain β_1, then $\mathcal{A}^\Pi$ is just the product of as many copies of $\mathcal{A}$ as Π has elements. Hence (2) is clear. On the other hand, if Π contains β_1 as a (necessarily separated) subset, then by Lemma 1.1 we have

$$\text{gl. dim. } \mathcal{A}^\Pi \geqslant \text{gl. dim. } \mathcal{A}^{\beta_1} = 1 + \text{gl. dim. } \mathcal{A}.$$

THEOREM 4.2. *If $\mathcal{A}$ is nontrivial, then*

$$\text{gl.dim. } \mathcal{A}^\Pi = 1 + \text{gl.dim. } \mathcal{A} \tag{3}$$

if and only if Π contains β_1 but not β_2 as an ordered subset.

Proof. Suppose that Π contains β_1 but not β_2. By Lemma 4.1 the left side of (3) is greater than or equal to the right side. To prove the other inequality, by Corollary 3.2 it suffices to assume that Π is initial. But in this case the condition that Π does not contain β_2 is equivalent to the condition that every element has rank $\leqslant 1$. Hence the conclusion follows from Lemma 2.1.

Now suppose that (3) holds. Then again by Lemma 4.1, Π must contain β_1. If Π also contains β_2, then let $j \in \Pi$ be minimal with respect to the property that there exist elements i, p, q, satisfying $i < p < j$ and $i < q < j$ with p and q incompatible. Now Π_j is a coseparated subset of Π, and so by Corollary 1.2 we have

$$\text{gl. dim. } \mathcal{A}^{\Pi_j} \leqslant \text{gl. dim. } \mathcal{A}^\Pi \tag{4}$$

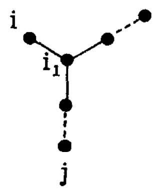
Also let Π' be the ordered subset of Π_j consisting of the four elements i, j, p, q. If Π' were not a separated subset of Π_j, then we would have an element $j' \in \Pi_j - \Pi'$ such that $p < j'$ and $q < j'$. But since $j' \in \Pi_j$, we have $j' < j$, contradicting the minimality of j. Hence Π' is a separated subset of Π_j and so combining Lemma 1.1, (3) and (4), we obtain

$$\text{gl. dim. } \mathcal{A}^{\Pi'} \leqslant 1 + \text{gl. dim. } \mathcal{A}.$$

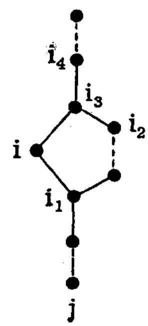
Since Π' is β_2, this is a contradiction.

Before characterizing sets of dimension 2, we prove some lemmas concerned only with finite ordered sets. We shall call an element i *shallow* in Π if Π_i is linearly ordered, and we shall say that Π is *well covered* if all shallow elements have at least two covers. Also we shall say that Π is *almost well covered* if there is precisely one shallow element with fewer than two covers. If $i < j$, we shall say that j *counts on* i if j is nonshallow with at most one cover, and j becomes shallow when i is removed from Π. In other words, j counts on i if and only if j has at most one cover, and Π_j consists of i together with a linearly-ordered subset which has at least one element incompatible with i.

Thus, depending on whether or not i is minimal, Π_j looks like one of the following:



(6)



(7)

It may be that $i_1 = j$, or in other words that j is a cover for i. It may also be that i_3 is minimal (in which case there is no i_4).

LEMMA 4.3. *If Π is any finite ordered set and j counts on i but is not a cover for i, then Π and $\Pi - \{j\}$ have the same number of shallow elements with fewer than two covers.*

Proof. Removal of j cannot create any new shallow elements since j is nonshallow. Also since j is the cover of only one element [Figs. (6) and (7)] and since this element is nonshallow, it follows that no shallow element loses any covers by removal of j.

LEMMA 4.4. *If Π is almost well covered, then there is an element $k \in \Pi$ such that $\Pi - \{k\}$ is either well covered or almost well covered. Furthermore we may assume that k has fewer than 2 cocovers.*

Proof. Let i be the unique shallow element which has at most one cover. If nothing counts on i, then since i is the cover of at most one element, there will be at most one shallow element which has fewer than two covers after removal of i. Thus we may assume that there is some element which counts on i, and by Lemma 4.3 we may assume, furthermore, that the only elements which count on i are its covers. Consequently, since i has only one cover i_1, we see that i_1 is the only element which counts on i. If i is minimal, then from Fig. (6) we see that removal of i leaves only i_1 as a shallow element with fewer than 2 covers. If i is not minimal, then we refer to Fig. (7). If i_3 has another cover besides i and i_2, then we can again remove i leaving only i_1 as a shallow element with fewer than two covers. Thus suppose that i_3 has only i and i_2 as covers. Again by Lemma 4.3 we may assume that the only elements counting on i_3 are covers for it, and since its only covers are the shallow elements i and i_2, it follows that i_3 has nothing counting on it. Also in the case where i_3 is not minimal, we see that after removal of i_3, i_2 and i will become covers for i_4. Thus we may remove i_3 and leave only i as a shallow element with fewer than two covers.

The second statement in the lemma follows from examining the various cases used to prove the first statement, and observing that in each case the element removed has fewer than 2 cocovers.

LEMMA 4.5. *If Π is well covered or almost well covered and has at least two elements, then Π contains an uncrossed crown.*

Proof. By induction on the number of elements of Π. The lemma is true vacuously when Π has two elements. Thus assume that Π has more than two elements, and that all well-covered or almost well-covered sets with one fewer element than Π contain uncrossed crowns. If Π is almost well covered, then by Lemma 4.4 we can remove a k so as to meet the hypothesis, and thus $\Pi - \{k\}$ contains a crown. Furthermore, since k has fewer than two cocovers, it follows that the crown has no crosspoint in Π if it had none in $\Pi - \{k\}$.

Hence assume that Π is well covered. If Π has a minimal element i with no more than one element counting on it, then we can simply apply our induction hypothesis to $\Pi - \{i\}$. Thus we may assume that each minimal element i has at least two elements counting on it, and furthermore by Lemma 4.3 we may assume that the elements which count on it are all covers for it. Let I be the set of all minimal elements, and let J be the set of all elements j such that j counts on and is a cover for some member of I. Then by assumption, each member of I has at least two members of J following it, and by definition of the term counts on, each member of J has precisely two members of I preceding it. If k is a crosspoint, say $i_1 \leqslant k, i_2 \leqslant k, j_1 \geqslant k, j_2 \geqslant k$ where i_1 and i_2 are distinct members of I, and j_1 and j_2 are distinct members of J, then since j_1 is a cover for either i_1 or i_2 and i_1 and i_2 are incompatible, this forces $k = j_1$. Similarly $k = j_2$, contradicting the fact that j_1 and j_2 are distinct. This completes the proof of the lemma.

THEOREM 4.6. *If $\mathfrak{A}$ is nontrivial, then*

$$\text{gl.dim. } \mathfrak{A}^{\Pi} = 2 + \text{gl.dim. } \mathfrak{A} \tag{8}$$

if and only if Π contains β_2, but no uncrossed, suspended crown.

Proof. Suppose that Π contains β_2, but no uncrossed, suspended crown. By Theorem 4.2, the left side of (8) is greater than or equal to the right side. To prove the other inequality, by Corollary 3.2 it suffices to assume that Π is both initial and terminal. But in this case the condition that Π contains no uncrossed, suspended crown is equivalent to the condition that Π contains no uncrossed crown. Therefore by Lemma 4.3 we may assume that Π has an element j of rank $\leqslant 1$ such that j has only one cover k. Let $\Pi' = \Pi - \{j\}$, and let $F : \mathfrak{A}^\Pi \to \mathfrak{A}^{\Pi'}$ and $G : \mathfrak{A}^{\Pi'} \to \mathfrak{A}^\Pi$ be as usual. Notice that Π' is separated since j has rank $\leqslant 1$ and Π is initial. Let S_i and S_i' denote the coadjoints for the evaluation functors relative to $\mathfrak{A}^\Pi$ and $\mathfrak{A}^{\Pi'}$ respectively. Then we have

$$FS_i = S_i' \qquad \text{for} \quad i \neq j \tag{9}$$

$$FS_j = S_k' . \tag{10}$$

Eq. (10) results from the fact that k is the only cover of j, and is, in fact, the whole reason for Lemma 4.5. Suppose that gl.dim. $\mathfrak{A} = n$, and that $\mathfrak{A}^\Pi$ has an object D of homological dimension $n + 3$. As in Section 1, we can find exact sequences in $\mathfrak{A}^\Pi$

$$0 \to K^1 \to \bigoplus_{i \in \Pi} S_i(D_i) \to D \to 0 \tag{11}$$

$$0 \to K^2 \to \bigoplus_{i \in \Pi} S_i(L_i^1) \to K^1 \to 0 \tag{12}$$

where $K_i^2 = 0$ for all elements i of rank 0 or 1. It follows that $GF(K^2) = K^2$, and consequently $F(K^2)$ and K^2 have the same homological dimension. But since all objects in $\mathfrak{A}$ have homological dimension $\leqslant n$ and $\text{h.d.}_\Pi D = n + 3$, it follows from (11) and (12) that $\text{h.d.}_\Pi K^2 = n + 1$. Consequently $\text{h.d.}_{\Pi'} F(K^2) = n + 1$. Applying F to (11) and (12) and using (9) and (10), we then see that $\text{h.d.}_{\Pi'} F(D) = n + 3$. But since Π' has one fewer element than Π, we may assume by induction that

$$\text{gl. dim. } \mathfrak{A}^{\Pi'} \leqslant 2 + \text{gl. dim. } \mathfrak{A}.$$

This contradiction proves the theorem in one direction.

Now suppose that (8) holds. Then by Theorem 4.2, Π must contain β_2. Suppose that Π also contains an uncrossed, suspended crown I, J. Let Π'' be the ordered subset of Π consisting of the elements of I and J, together with an element preceding all elements of I and one following all elements in J. Also let Π' denote Π'', together with all members of Π which follow no more than one member of I. Then Π'' is clearly a separated subset of Π'. We claim that Π' is a coseparated subset of Π. For if $i \notin \Pi'$, then any pair of incompatible elements of Π' following i must be in J. But since i follows a pair in I, this contradicts the fact that the crown is uncrossed. We can now apply Lemma 1.1, Corollary 1.2, and Theorem 3.8 to obtain

$$\text{gl. dim. } \mathfrak{A}^\Pi \geqslant \text{gl. dim. } \mathfrak{A}^{\Pi'} \geqslant \text{gl. dim. } \mathfrak{A}^{\Pi''} = 3 + \text{gl. dim. } \mathfrak{A}.$$

This contradiction completes the proof of the theorem.

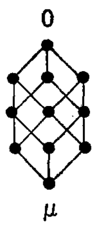
Conjecture. Let Π have unique minimal and maximal elements 0 and μ respectively. If μ has depth 0 (that is, if Π consists of a single element) then we say that Π is 0-special. Inductively we say that Π is ***m*-special** if

(1) μ is of depth m,

(2) Π_i is $m - 1$ special for all cocovers i of μ, and

(3) each element of depth $m - 2$ in Π has at least two covers.

In particular Π is 1-special if and only if it is a linearly-ordered set of two elements. A set is 2-special if and only if it consists of a suspended collection of mutually incompatible elements. The 3-special sets are precisely the uncrossed, suspended crowns. A generalizzd m-braid is m-special. The following set contains 3-special subsets, but is not 4-special:



(13)

It is not difficult to see that if Π is m-special, then so is Π^*, and if Π_1 and Π_2 are m_1 and m_2-special respectively, then $\Pi_1 \times \Pi_2$ is $m_1 + m_2$-special. Also if Π is m-special and $0 < k < m$, then the set obtained by removal of all elements of depth k is $(m - 1)$-special. It should be possible to prove that if Π is m-special, then $L_0(Z)$ is a strong Z-muscle of homological dimension m. The dimension of a general finite ordered set Π might then be the maximum m such that Π contains an m-special set, or some variety thereof, as some kind of ordered subset. In any case, if one writes down a projective resolution for $L_0(Z)$ relative to diagram (13), one finds that the homological dimension of this object is 3, not 4.

5. Relative Homological Dimension

Following Buchsbaum [*1*], [*2*], and using the terminology of MacLane [*10*], we define a *proper class* $\mathscr{C}$ of short exact sequences in an Abelian category as one satisfying the following self dual set of axioms, where we let $\mathscr{M}(\mathscr{C})$ and $\mathscr{E}(\mathscr{C})$ denote the classes of monomorphisms and epimorphisms respectively associated with short exact sequences in $\mathscr{C}$.

(i) All coretracts (split monomorphisms) are in $\mathscr{M}(\mathscr{C})$ [and consequently all retracts are in $\mathscr{E}(\mathscr{C})$].

(ii) If $\alpha, \beta \in \mathscr{M}(\mathscr{C})$ and $\beta\alpha$ is defined, then $\beta\alpha \in \mathscr{M}(\mathscr{C})$.

(ii*) If $\gamma, \delta \in \mathscr{E}(\mathscr{C})$ and $\delta\gamma$ is defined, then $\delta\gamma \in \mathscr{E}(\mathscr{C})$.

(iii) If $\beta\alpha \in \mathscr{M}(\mathscr{C})$ and β is a monomorphism, then $\alpha \in \mathscr{M}(\mathscr{C})$.

(iii*) If $\delta\gamma \in \mathscr{E}(\mathscr{C})$ and γ is an epimorphism, then $\delta \in \mathscr{E}(\mathscr{C})$.

It is not difficult to show (see for example [*11*, p. 188, Exercise 6]) that axioms (i), (ii*), and (iii*) imply the following:

(iv) If $E \in \mathscr{C}$ and αE is defined, then $\alpha E \in \mathscr{C}$.

Condition (iv) is easily seen to be equivalent to the following strengthening of (iii*) : If $\delta\gamma \in \mathscr{E}(\mathscr{C})$, then $\delta \in \mathscr{E}(\mathscr{C})$ [γ not necessarily an epimorphism]. By duality axioms (i), (ii), and (iii) imply the following strengthening of (iii):

(iv*) If $E \in \mathscr{C}$ and $E\gamma$ is defined, then $E\gamma \in \mathscr{C}$.

Conversely, in [6] Freyd showed that axioms (i), (ii), (iii), and (iv) imply axiom (ii*) and thereby established the equivalence of the former set of axioms with the set (i), (ii*), (iii*), and (iv*). However the proof given in [6] utilizes functor categories, which therefore limits its validity to small categories. We give here a more elementary proof, which is valid for all Abelian categories. The problem is equivalent to establishing the following lemma:

LEMMA 5.1. *Consider the following exact, commutative diagram*

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, equal] & B' \arrow[r, "\gamma"] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\delta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& & C'' \arrow[r, equal] \arrow[d] & C'' \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

where α *and* τ *are in* $\mathscr{M}(\mathscr{C})$. *If* $\mathscr{C}$ *satisfies* (i), (ii), (iii), and (iv), *then* $\beta \in \mathscr{M}(\mathscr{C})$.

Proof. Consider the pushout diagram

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & B' \arrow[d, "\nu"] \\
B \arrow[r, "\pi"] & D
\end{tikzcd}$$

By (iv) we have $\nu \in \mathcal{M}(\mathscr{C})$. Also we have

$$(\pi\beta - \nu)\sigma = \pi\alpha - \nu\sigma = 0,$$

and so there is a morphism $\lambda : C' \to D$ such that

$$\lambda\gamma = \pi\beta - \nu. \tag{1}$$

Now form the pushout diagram

$$\begin{tikzcd} C' \arrow[r, "\lambda"] \arrow[d, "\tau"'] & D \arrow[d, "\kappa"] \\ C \arrow[r, "\mu"] & E \end{tikzcd}$$

Again by (iv) we have $\kappa \in \mathcal{M}(\mathscr{C})$. Then we have, using (1),

$$\mu\delta\beta = \mu\tau\gamma = \kappa\lambda\gamma = \kappa(\pi\beta - \nu),$$

or rewriting,

$$(\kappa\pi - \mu\delta)\beta = \kappa\nu.$$

But since both ν and $\kappa \in \mathcal{M}(\mathscr{C})$, by (ii) we have $\kappa\nu \in \mathcal{M}(\mathscr{C})$. Therefore (iv*) enables us to conclude that $\beta \in \mathcal{M}(\mathscr{C})$.

In general it is easier to verify axioms (i)–(iv), or the equivalent dual set of axioms (i*)–(iv*), then it is to verify the original set (i), (ii), (iii), (ii*), (iii*). For example, cohsider a kernel preserving functor $T : \mathfrak{A} \to \mathscr{B}$ and let $\mathscr{D}$ be a proper class in $\mathscr{B}$. Let $T^{-1}(\mathscr{D})$ be the class of all those short exact sequences in $\mathfrak{A}$ which are carried by T into a member of $\mathscr{D}$. Then taking into account the fact that kernel preserving functors preserve pullbacks [*11*, p. 56, Corollary 6.7] it is trivial to show that $T^{-1}(\mathscr{D})$ satisfies (i*)–(iv*). By duality, if T is cokernel preserving, then $T^{-1}(\mathscr{D})$ satisfies (i)–(iv).

If $\mathscr{C}$ is a proper class in $\mathfrak{A}$, then we can form the relative extension functors $\mathrm{Ext}_{\mathscr{C}}^n(A, C)$, where again $\mathrm{Ext}_{\mathscr{C}}^0(A, C) = [A, C]$ (see [*1*], [*6*], [*10*, p. 369], or [*11*, p. 188, Exercise 6]). We can then define $\mathrm{h.d.}_{\mathscr{C}}\, A$ as the largest integer n (or infinity) for which the one variable functor $\mathrm{Ext}_{\mathscr{C}}^n(A, \quad)$ is not zero, and $\mathrm{gl.dim.}_{\mathscr{C}}\, \mathfrak{A}$ as the sup of the $\mathrm{h.d.}_{\mathscr{C}}\, A$. The proof of the crucial Lemma 1.1 of [*12*] can then be repeated to give the following:

LEMMA 5.2. *Let $S : \mathscr{B} \to \mathfrak{A}$ be a coadjoint for $T : \mathfrak{A} \to \mathscr{B}$, and suppose that $\mathscr{C}$ and $\mathscr{D}$ are proper classes in $\mathfrak{A}$ and $\mathscr{B}$ respectively such that $\mathscr{C} \subset T^{-1}(\mathscr{D})$ and $\mathscr{D} \subset S^{-1}(\mathscr{C})$. Then we have natural equivalences*

$$\mathrm{Ext}_{\mathscr{C}}^n(S(B), A) \approx \mathrm{Ext}_{\mathscr{D}}^n(B, T(A))$$

for all $n \geqslant 0$.

The following proposition shows how the situation in Lemma 5.2 can be manufactured.

Proposition 5.3. *Let $S : \mathscr{B} \to \mathfrak{A}$ be a coadjoint for a faithful functor $T : \mathfrak{A} \to \mathscr{B}$, and let $\mathscr{D}'$ be a proper class in $\mathscr{B}$. Then if either S or T is full, we have*

$$T^{-1}(S^{-1}(T^{-1}(\mathscr{D}'))) = T^{-1}(\mathscr{D}').$$

Consequently if we take $\mathscr{C} = T^{-1}(\mathscr{D}')$ and $\mathscr{D} = S^{-1}(T^{-1}(\mathscr{D}')$, then $\mathscr{C} = T^{-1}(\mathscr{D})$ and $\mathscr{D} = S^{-1}(\mathscr{C})$.

Proof. Fullness of S or T implies that TST is naturally equivalent to T [*11*, p. 120, Proposition 1.3]. Now if

$$O \to A' \to A \to A'' \to O \tag{2}$$

is in $T^{-1}(S^{-1}(T^{-1}(\mathscr{D}')))$, then

$$O \to TST(A') \to TST(A) \to TST(A'') \to O \tag{3}$$

is in $\mathscr{D}'$, and consequently

$$O \to T(A') \to T(A) \to T(A'') \to O \tag{4}$$

is in $\mathscr{D}'$. In other words (2) is in $T^{-1}(\mathscr{D}')$.

Conversely if (2) is in $T^{-1}(\mathscr{D}')$, then (4) is in $\mathscr{D}'$, and consequently (3) is in $\mathscr{D}'$. In particular, this means that (3) is exact, and so by faithfulness of T,

$$O \to ST(A) \to ST(A) \to ST(A'') \to O \tag{5}$$

is exact. Hence (5) is in $T^{-1}(\mathscr{D}')$, and so (4) is in $S^{-1}T^{-1}(\mathscr{D}')$. In other words (2) is in $T^{-1}(S^{-1}(T^{-1}(\mathscr{D}')))$ as required.

If Π is a small category and $\mathscr{D}$ is a proper class in $\mathfrak{A}$, then

$$\bigcap_{i \in \Pi} T_i^{-1}(\mathscr{D}) = \mathscr{C}$$

is a proper class in $\mathfrak{A}^{\Pi}$, where T_i represents the evaluation functor corresponding to the object $i \in \Pi$. It is routine to state and prove counterparts relative to $\mathscr{C}$ and $\mathscr{D}$ for the theorems of [*12*] and Sections 1–4 of the present paper. The important thing to be observed is that the short exact sequences in $\mathfrak{A}^{\Pi}$ which we have made use of always split on application of the evaluation functors, and hence are necessarily in $\mathscr{C}$. As examples, we have

Theorem 5.4. *Let $\mathscr{D}$ be a proper class of exact sequences in $\mathfrak{A}$, and let Π be a partially-free monoid on I generators. If $\mathfrak{A}$ has products or coproducts indexed over the maximum of the cardinal numbers of I and the integers, and if furthermore $\mathscr{D}$ is closed to such products or coproducts, then*

$$\text{gl.dim.}_{\mathscr{C}}\, \mathfrak{A}^{\Pi} = 1 + \text{gl.dim.}_{\mathscr{D}}\, \mathfrak{A}$$

where $\mathscr{C} = T^{-1}(\mathscr{D})$.

THEOREM 5.5. *Let $\mathscr{D}$ be a proper class in $\mathfrak{A}$, and let*

$$\mathscr{C} = \bigcap_{i \in \Pi} T_i^{-1}(\mathscr{D}).$$

If Π is a generalized m-braid, then

$$\text{gl.dim.}_{\mathscr{C}}\, \mathfrak{A}^{\Pi} = m + \text{gl.dim.}_{\mathscr{D}}\, \mathfrak{A}.$$

If Π is an uncrossed, suspended crown, then

$$\text{gl.dim.}_{\mathscr{C}}\, \mathfrak{A}^{\Pi} = 3 + \text{gl.dim.}_{\mathscr{D}}\, \mathfrak{A}.$$

6. EXT-PRESERVING FUNCTORS

Let $\mathscr{C}$ and $\mathscr{D}$ be proper classes in categories $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We shall say that a functor $T : \mathfrak{A} \to \mathfrak{B}$ has property (P) with respect to $\mathscr{C}$ and $\mathscr{D}$ if any situation of the form

$$\begin{tikzcd} & T(A'') \arrow[d] \\ B \arrow[r, "\beta"] & B'' \end{tikzcd}$$

where $\beta \in \mathscr{E}(\mathscr{D})$ can be extended to a commutative diagram

$$\begin{tikzcd} T(A) \arrow[r, "T(\alpha)"] \arrow[d] & T(A'') \arrow[d] \\ B \arrow[r, "\beta"] & B'' \end{tikzcd}$$

where $\alpha \in \mathscr{E}(\mathscr{C})$. We say that T has property (P^*) if the functor obtained by composing T with the duality functors on $\mathfrak{A}$ and $\mathfrak{B}$ has property (P).

The functor T is called *Ext preserving with respect to $\mathscr{C}$ and $\mathscr{D}$* if $\mathscr{C} \subset T^{-1}(\mathscr{D})$, and if the natural transformation thereby induced by T

$$\theta : \text{Ext}_{\mathscr{C}}{}^n(A'', A') \to \text{Ext}_{\mathscr{D}}{}^n(T(A''), T(A))$$

is a natural equivalence for all $n \geqslant 0$. Taking $n = 0$, we see that if T is Ext preserving with respect to $\mathscr{C}$ and $\mathscr{D}$, then T must be full and faithful. If $\mathscr{C}$ and $\mathscr{D}$ are the classes of all exact sequences in $\mathfrak{A}$ and $\mathfrak{B}$, then we shall simply say that T is *Ext preserving*.

THEOREM 6.1. *Let T be a full and faithful functor such that $\mathscr{C} \subset T^{-1}(\mathscr{D})$ and such that T has either property (P) or (P^*) with respect to proper classes $\mathscr{C}$*

and $\mathscr{D}$ in $\mathfrak{A}$ and $\mathscr{B}$ respectively. Then T is Ext preserving with respect to $\mathscr{C}$ and $\mathscr{D}$, and furthermore $\mathscr{C} = T^{-1}(\mathscr{D})$.

Proof. Fullness and faithfulness of T give the result for $n = 0$. Also the case where T has property (P^*) follows by duality from the case where T has property (P), and so we shall assume the latter. We first prove a lemma, which amounts to showing that θ is an isomorphism for $n = 1$.

LEMMA 6.2. *Let T be as in Theorem 6.1. Then given a member of $\mathscr{D}$ of the form*

$$O \to T(A') \to B \to T(A'') \to O, \tag{1}$$

there is a member E of $\mathscr{C}$ such that (1) is equivalent to $T(E)$.

Proof. Using (P), we obtain a pushout diagram

$$\begin{tikzcd} O \arrow[r] & T(K) \arrow[r] \arrow[d] & T(A) \arrow[r] \arrow[d] & T(A'') \arrow[r] \arrow[d, equal] & O \\ O \arrow[r] & T(A') \arrow[r] & B \arrow[r] & T(A'') \arrow[r] & O \end{tikzcd}$$

where $O \to K \to A \to A'' \to O$ is in $\mathscr{C}$. Forming the pushout diagram in $\mathfrak{A}$

$$\begin{tikzcd} O \arrow[r] & K \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d, equal] & O \\ O \arrow[r] & A' \arrow[r] & \bar{A} \arrow[r] & A'' \arrow[r] & O \end{tikzcd} \tag{2}$$

we know the bottom row E is in $\mathscr{C}$ by axiom (iv) for proper classes. Applying T to (2) and using the fact that $T(\mathscr{C}) \subset \mathscr{D}$, we obtain another pushout diagram. Consequently our result follows from the uniqueness of pushouts [*11*, p. 163, Corollary 1.2*].

It now follows by replacing B by an object of the form $T(A)$ in Lemma 6.2 and using fullness and faithfulness of T that $\mathscr{C} = T^{-1}(\mathscr{D})$.

Now let F represent an element of $\mathrm{Ext}_{\mathscr{D}}{}^n(T(A''), T(A'))(n \geqslant 1)$. This means that we can decompose F into short exact sequences

$$O \to M_{i+1} \to B_i \to M_i \to O$$

in $\mathscr{D}$ for $0 \leqslant i \leqslant n-1$, where $M_0 = T(A'')$ and $M_n = T(A')$. Using (P) we can construct successively

$$\begin{tikzcd} O \arrow[r] & T(K_{i+1}) \arrow[r] \arrow[d] & T(A_i) \arrow[r] \arrow[d] & T(K_i) \arrow[r] \arrow[d] & O \\ O \arrow[r] & M_{i+1} \arrow[r] & B_i \arrow[r] & M_i \arrow[r] & O \end{tikzcd} \tag{3}$$

for $0 \leqslant i \leqslant n-2$, where $K_0 = A''$ and $T(K_0) \to M_0$ is the identity on $T(A'')$. Also form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \to & T(A') & \to & B & \to & T(K_{n-1}) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & T(A') & \to & B_{n-1} & \to & M_{n-1} & \to & 0 \end{array} \tag{4}$$

Then by Lemma 5.2 we can write $B = T(A_{n-1})$, and so the diagrams (3) can be combined with (4) to give us a morphism of sequences with fixed ends

$$T(\boldsymbol{E}) \to \boldsymbol{F} \tag{5}$$

where $\boldsymbol{E} \in \operatorname{Ext}_{\mathscr{C}}{}^n(A'', A')$. This shows that θ is onto. Since θ is additive, it suffices now to show that if $\boldsymbol{G} \in \operatorname{Ext}^n(A'', A')$ is such that $T(\boldsymbol{G}) \sim 0$, then $G \sim 0$. If $T(\boldsymbol{G}) \sim 0$, then by [*11*, p. 177, Theorem 4.2] we can find a morphism of sequences

$$\boldsymbol{F} \to T(\boldsymbol{G}) \tag{6}$$

with fixed ends, where $\boldsymbol{F}$ has a split monomorphism at its left end. Forming $\boldsymbol{E}$ as in (5), we see that this last property must also be true of $T(\boldsymbol{E})$. Composing (5) and (6) we get a morphism $T(\boldsymbol{E}) \to T(\boldsymbol{G})$ with fixed ends, and then using fullness and faithfulness of T, we obtain a morphism $\boldsymbol{E} \to \boldsymbol{G}$ with fixed ends, where $\boldsymbol{E}$ splits at its left end. This shows that $\boldsymbol{G} \sim 0$, and completes the proof of the theorem.

We now examine the case where $T : \mathfrak{A} \to \mathscr{B}$ has a coadjoint $S : \mathscr{B} \to \mathfrak{A}$. First we prove a well-known proposition concerning adjoint functors.

PROPOSITION 6.3. *If $T : \mathfrak{A} \to \mathscr{B}$ has a coadjoint $S : \mathscr{B} \to \mathfrak{A}$, then the following two conditions are equivalent*:

(a) *T is full and faithful.*

(b) *$\psi : ST \to 1_{\mathfrak{A}}$ is a natural equivalence. These two conditions are implied by the condition*

(c) *T is faithful and S is full.*

Proof. For fixed A consider the composition of natural transformations of functors of A'

$$[A, A'] \to [ST(A), A'] \approx [T(A), T(A')]$$

where the first one is induced by ψ_A and the second is the natural equivalence giving adjointness. If $\alpha \in [A, A']$, then it is easily seen that α is carried into $T(\alpha)$ by this composition. Hence T is full and faithful if and only if the composition is a natural equivalence, and the latter is true if and only if ψ_A is an isomorphism.

Now suppose that S is full and T is faithful. Then since S is full, $T(\psi_A)$ is an isomorphism by [*11*, p. 120, Proposition 1.3]. Hence since T is faithful, ψ_A is an isomorphism.

Remark. The above proof is valid for non-Abelian categories $\mathcal{A}$ and $\mathcal{B}$, providing we assume in condition (c) that $\mathcal{A}$ is balanced.

PROPOSITION 6.4. *Let $S : \mathcal{B} \to \mathcal{A}$ be a coadjoint for $T : \mathcal{A} \to \mathcal{B}$, and let $\mathcal{C}$ and $\mathcal{D}$ be proper classes in $\mathcal{A}$ and $\mathcal{B}$ respectively. If T has property (P^*), then $\mathcal{D} \subset S^{-1}(\mathcal{C})$. Conversely if T is faithful and either S or T is full, and if $\mathcal{D} \subset S^{-1}(\mathcal{C})$, then T has property (P^*).*

Proof. Suppose that T has property (P^*), and let $\beta : B' \to B$ be in $\mathcal{M}(\mathcal{D})$. Then by (P^*) we can form a commutative diagram

$$\begin{tikzcd} B' \arrow[r, "\beta"] \arrow[d, "\phi_{B'}"'] & B \arrow[d] \\ TS(B') \arrow[r, "T(\alpha)"] & T(A) \end{tikzcd} \tag{7}$$

where $\alpha : S(B') \to A$ is in $\mathcal{M}(\mathcal{C})$. Applying S to (7) and using the natural transformation $\psi : ST \to 1_{\mathcal{A}}$, we obtain the commutative diagram

$$\begin{tikzcd} S(B') \arrow[r, "S(\beta)"] \arrow[d] & S(B) \arrow[d] \\ STS(B') \arrow[r] \arrow[d] & ST(A) \arrow[d] \\ S(B') \arrow[r, "\alpha"] & A \end{tikzcd}$$

Using the relation $\psi_{S(B')}S(\varphi_{B'}) = 1_{S(B')}$ and also axiom (iv*) for proper classes, we see that $S(\beta) \in \mathcal{C}$. This proves that $\mathcal{D} \subset S^{-1}(\mathcal{C})$.

Conversely, suppose that T is faithful and either S or T is full, and that $\mathcal{D} \subset S^{-1}(\mathcal{C})$. Given $\beta : B' \to B$ in $\mathcal{M}(\mathcal{D})$ and a morphism $B' \to T(A')$, we form the diagram

$$\begin{tikzcd} B' \arrow[r, "\beta"] \arrow[d] & B \arrow[d] \\ T(A') \arrow[r, "\gamma"] \arrow[d] & P \arrow[d] \\ TST(A') \arrow[r] & TS(P) \end{tikzcd} \tag{8}$$

where the upper square is a pushout and the lower square is induced by φ. By Proposition 6.3 we know that $\psi_{A'}$, is an isomorphism, and so let μ be its inverse. Then since $T(\psi_{A'})\varphi_{T(A')} = 1_{T(A')}$, it follows that $\varphi_{T(A')} = T(\mu)$. Now by axiom (iv) for proper classes, the morphism γ is in $\mathscr{M}(\mathscr{D})$. Hence by assumption on S, $S(\gamma)$ is in $\mathscr{M}(\mathscr{C})$, and so since μ is an isomorphism, $S(\gamma)\mu$ is in $\mathscr{M}(\mathscr{C})$. But then since the morphism $T(A') \to TS(P)$ in (8) is just $T(S(\gamma))T(\mu) = T(S(\gamma)\mu)$, this shows that T has property (P^*).

Combining Theorem 6.1 and Proposition 6.4 we obtain

COROLLARY 6.5. *Let $S : \mathscr{B} \to \mathfrak{A}$ be a coadjoint for $T : \mathfrak{A} \to \mathscr{B}$, and let $\mathscr{C}$ and $\mathscr{D}$ be proper classes in $\mathfrak{A}$ and $\mathscr{B}$ respectively. Suppose that T is full and faithful. If $\mathscr{C} \subset T^{-1}(\mathscr{D})$ and $\mathscr{D} \subset S^{-1}(\mathscr{C})$, then T is Ext preserving with respect to $\mathscr{C}$ and $\mathscr{D}$, and $\mathscr{C} = T^{-1}(\mathscr{D})$.*

Remark. Corollary 6.5 could also be obtained from Lemma 5.2 using the fact that in this case ψ is a natural equivalence.

APPLICATIONS

(1) Let $\mathscr{C}$ be any proper class in a category $\mathfrak{A}$, and let $\mathscr{L}_{\mathscr{C}}(\mathfrak{A}^*)$ denote the category of contravariant Abelian group-valued functors from $\mathfrak{A}$ which have the property that if $0 \to A' \to A \to A'' \to 0$ is in $\mathscr{C}$, then $0 \to T(A'') \to T(A) \to T(A')$ is an exact sequence of Abelian groups. It is shown in [6] that $\mathscr{L}_{\mathscr{C}}(\mathfrak{A}^*)$ is an Abelian category with enough injectives, and that the functor

$$H : \mathfrak{A} \to \mathscr{L}_{\mathscr{C}}(\mathfrak{A}^*)$$

which assigns the contravariant morphism functor H_A to the object A takes members of $\mathscr{C}$ into members of $\mathscr{D}$, where $\mathscr{D}$ is the class of all exact sequences in $\mathscr{L}_{\mathscr{C}}(\mathfrak{A}^*)$. (Proofs of these statements for the nonrelative case can be found in [*11*, pp. 150, 151].) It is also shown in [6] that H satisfies property (P). (The nonrelative proof can be found in [*11*, 160, Exercise 19].) Consequently by Theorem 6.1, H is Ext preserving with respect to $\mathscr{C}$ and $\mathscr{D}$, and furthermore $\mathscr{C} = H^{-1}(\mathscr{D})$. (Both of these statements are also contained in [6].)

(2) Let $\mathfrak{A}$ be a C_4(A.B.6) category [*11*, Chapter X] and let X be a topological space. Let $f : A \to X$ be the inclusion map of a subspace, and let

$$f_* : \mathscr{F}(A, \mathfrak{A}) \to \mathscr{F}(X, \mathfrak{A})$$

be the direct image functor, where $\mathscr{F}(X, \mathfrak{A})$ denotes the category of sheaves in $\mathfrak{A}$ over X. Then f_* has an exact coadjoint f^* (the induced sheaf functor), and furthermore f_* is full and faithful [*11*, p. 258, Lemma 8.1]. Furthermore if A is a closed subspace of X, then by examining exact sequences stalkwise, it is not difficult to show that f_* is exact. Consequently from Corollary 6.5 it follows that f_* is Ext preserving.

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